Computer Graphics

3 - Transformations

Yoonsang Lee
Hanyang University

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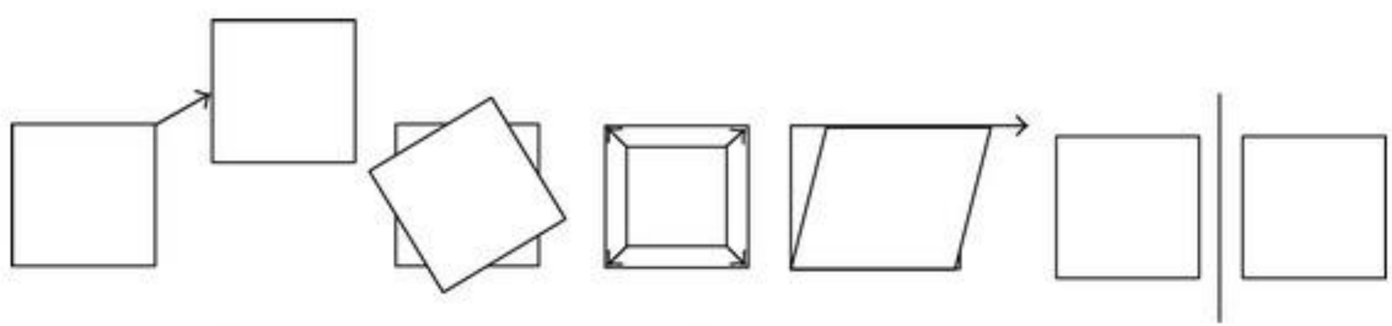
Outline

- 2D Transformations
  - Scaling, Rotation, Shearing, Reflection
  - Translation
- Classes of Transformations
- Composition of Transformations & Homogeneous Coordinates
- Two Types of 3D Cartesian Coordinate System
- 3D Affine Transformations

2D Transformations

What is Transformation?

- Geometric **Transformation**
  - The process of changing the position, orientation, size, or shape of a geometric object using mathematical operations.
    → "Moving a set of points"
  - Essential in computer graphics because it enables the creation of complex scenes and animations.
- Examples:

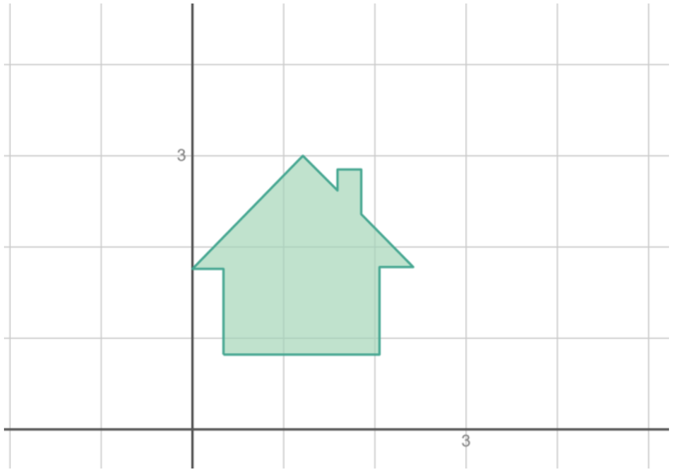


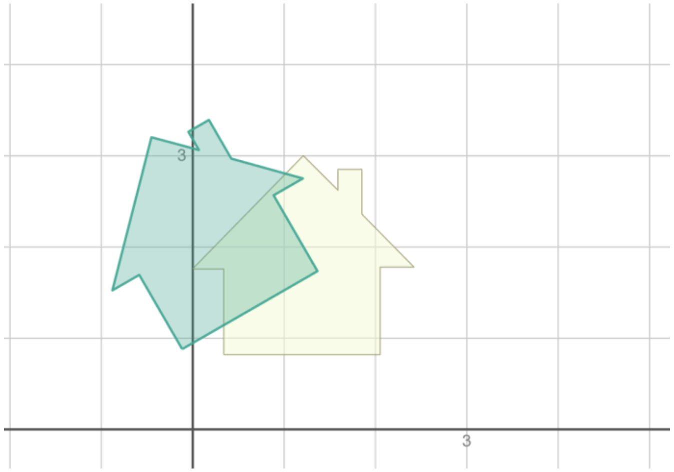
* This image is from the slides of Prof. Roger D. Eastman (University of Maryland): https://www.cs.umd.edu/~reastman/slides/L05P1Transformations.pdf

Transformation

- "Moving a set of points"
  - Transformation T maps any input vector v in the vector space S to T(v).

$$S \rightarrow \{T(\mathbf{v}) \mid \mathbf{v} \in S\}$$





Linear Transformation

- One way to define a transformation is by matrix multiplication:

$$T(\mathbf{v}) = M\mathbf{v}$$

- This is called a **linear transformation** because a matrix multiplication represents a linear mapping.

$$T(a\mathbf{u} + \mathbf{v}) = aT(\mathbf{u}) + T(\mathbf{v})$$

$$\mathbf{M} \cdot (a\mathbf{u} + \mathbf{v}) = a\mathbf{M}\mathbf{u} + \mathbf{M}\mathbf{v}$$

2D Linear Transformations

- 2x2 matrices represent 2D linear transformations such as:
  - uniform scaling
  - non-uniform scaling
  - rotation
  - shearing
  - reflection

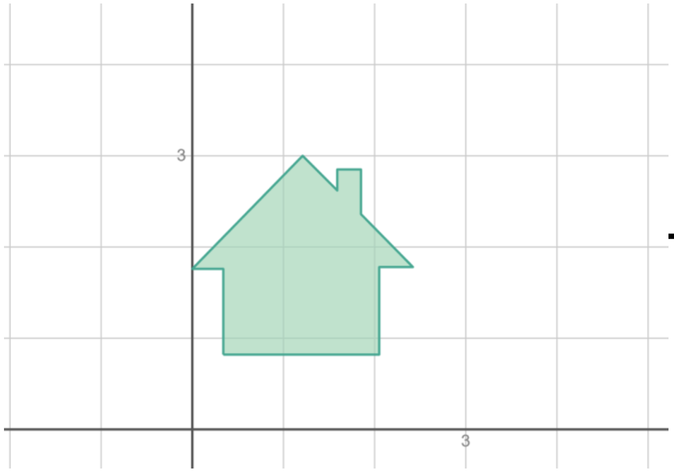
2D Linear Trans. – Uniform Scaling

- Uniformly shrinks or enlarges both in x and y directions.

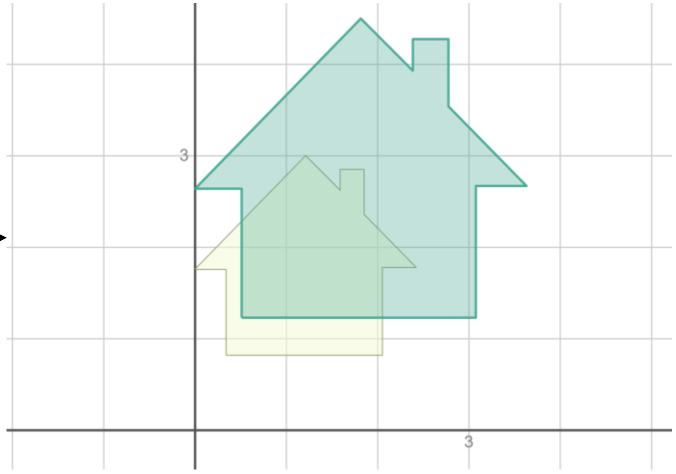
$$\begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} sx \\ sy \end{bmatrix}$$

2x2 scale matrix

S p = p'



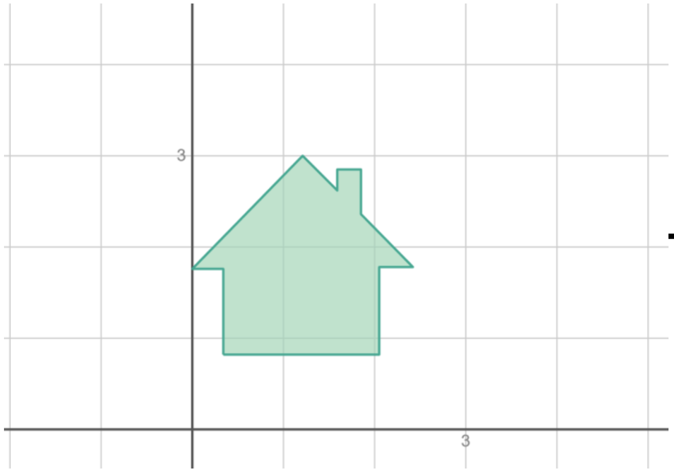
$$\begin{bmatrix} 1.5 & 0 \\ 0 & 1.5 \end{bmatrix}$$



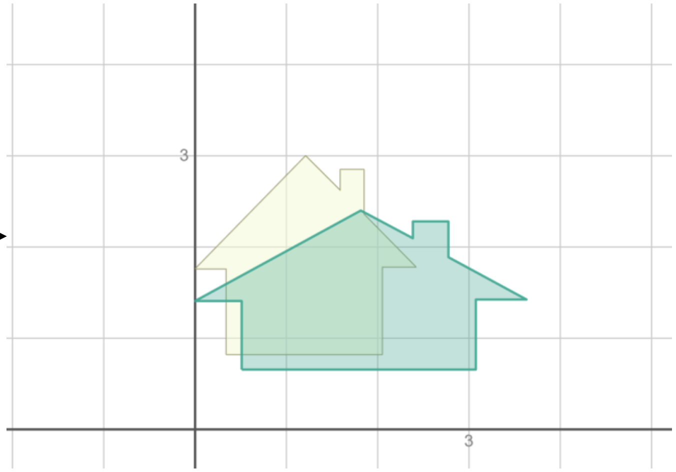
2D Linear Trans. – Nonuniform Scaling

- Non-uniformly shrinks or enlarges in x and y directions.

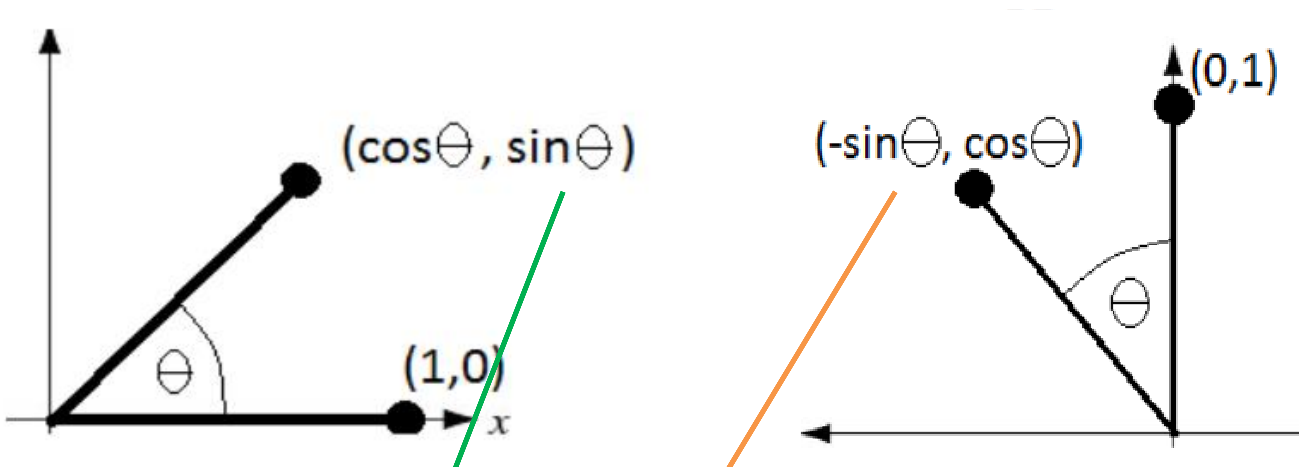
$$\begin{bmatrix} s_x & 0 \\ 0 & s_y \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} s_x x \\ s_y y \end{bmatrix}$$



$$\begin{bmatrix} 1.5 & 0 \\ 0 & 0.8 \end{bmatrix}$$



2D Linear Trans. – Rotation



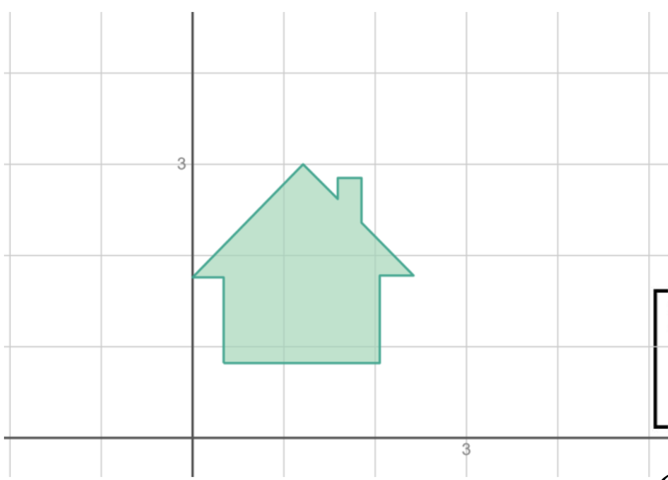
$$\Rightarrow R_\theta = \begin{bmatrix} \boxed{\begin{matrix}\cos\theta \\ \sin\theta\end{matrix}} & \boxed{\begin{matrix}-\sin\theta \\ \cos\theta\end{matrix}} \end{bmatrix}$$

: Rotation matrix

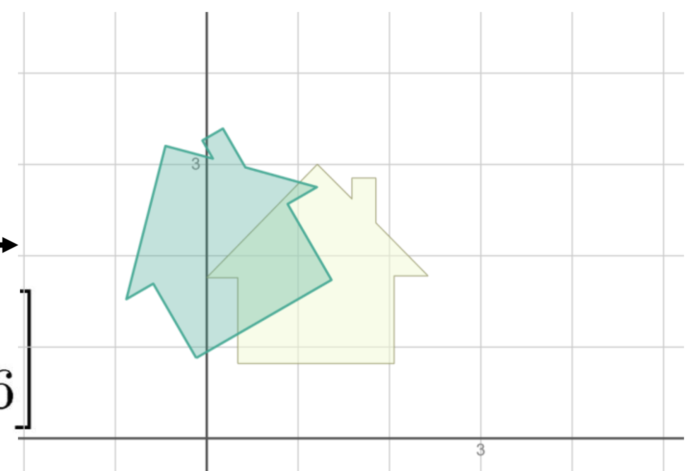
2D Linear Trans. – Rotation

- Rotation can be written in matrix multiplication, so it's also a linear transformation.
  - Note that positive angle means CCW rotation.

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x\cos\theta - y\sin\theta \\ x\sin\theta + y\cos\theta \end{bmatrix}$$



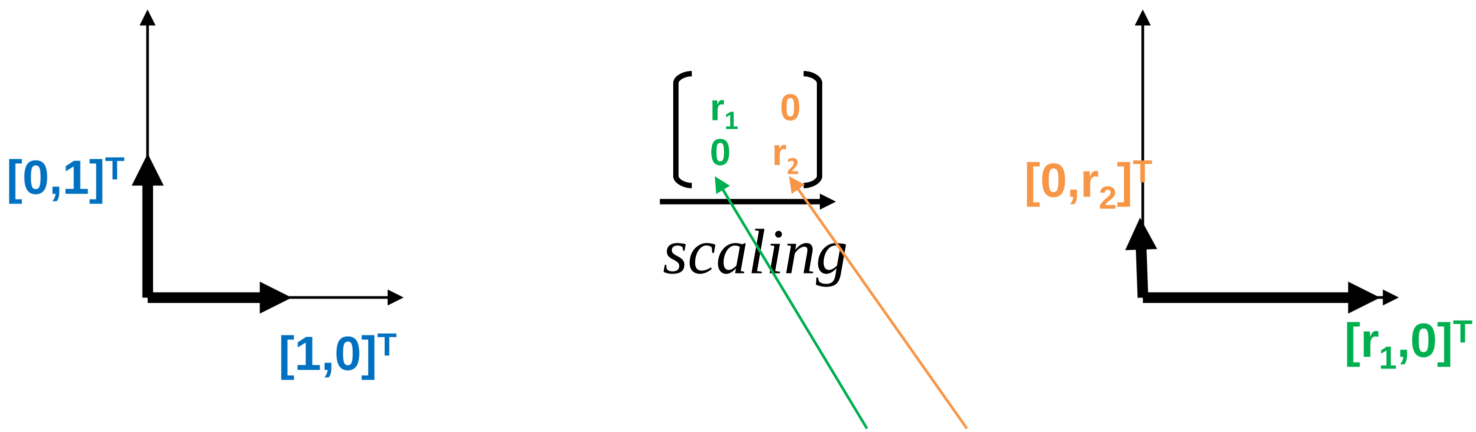
$$\begin{bmatrix} 0.866 & -0.5 \\ 0.5 & 0.866 \end{bmatrix}$$



(Rotate 30° CCW)

Numbers in Matrices: Scaling, Rotation

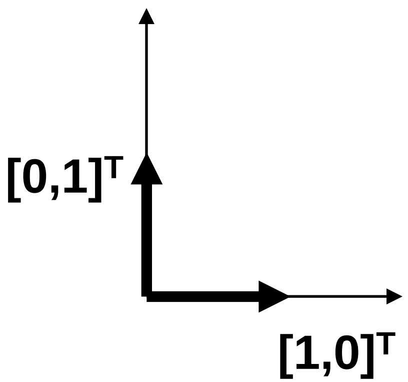
- Let’s think about what the numbers in the matrix means.



Canonical basis vectors: unit vectors pointing in the direction of the axes of a Cartesian coordinate system.

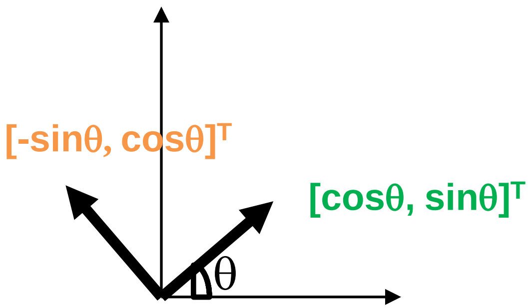
1st & 2nd basis vector of the transformed coordinates

Numbers in Matrices: Scaling, Rotation



$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

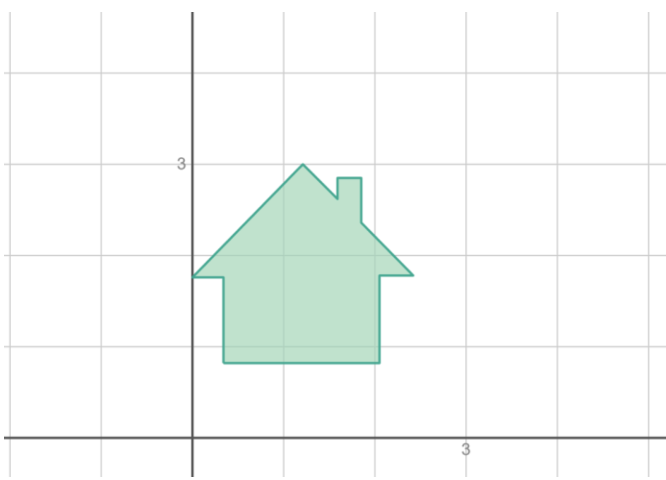
rotation



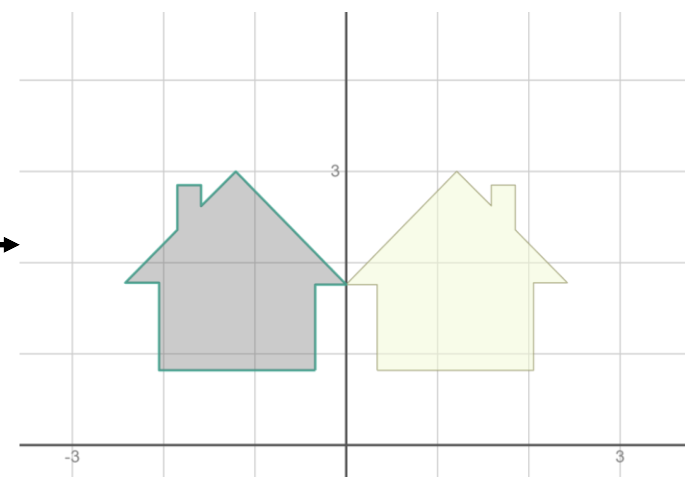
- *Column vectors* of a matrix is the *basis vectors of the column space (range)* of the matrix.
  - *Column space* of a matrix: The span (a set of all possible linear combinations) of its column vectors.

2D Linear Trans. – Reflection

- Reflection can be considered as a special case of non-uniform scale.



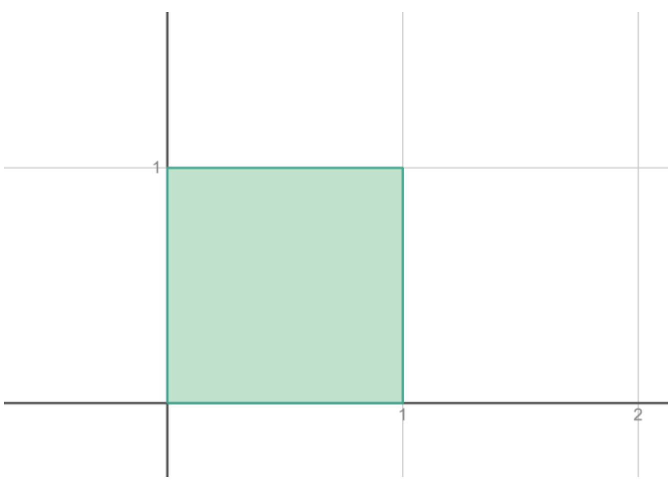
$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$



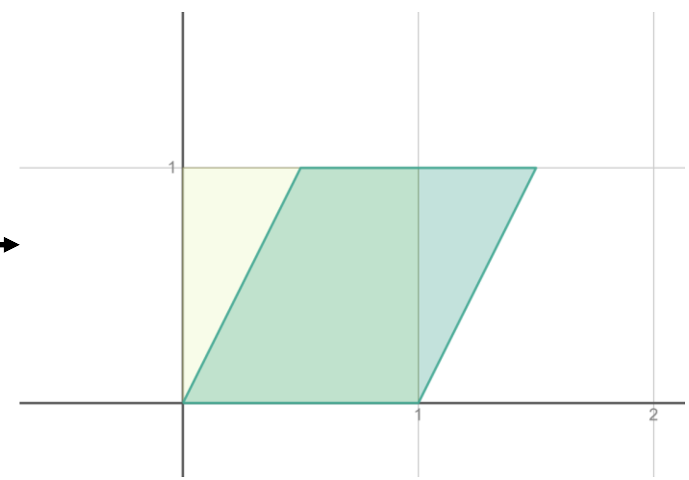
2D Linear Trans. – Shearing

- "Push things sideways"

$$\begin{bmatrix}1 & a\\0 & 1\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}x+ay\\y\end{bmatrix}$$



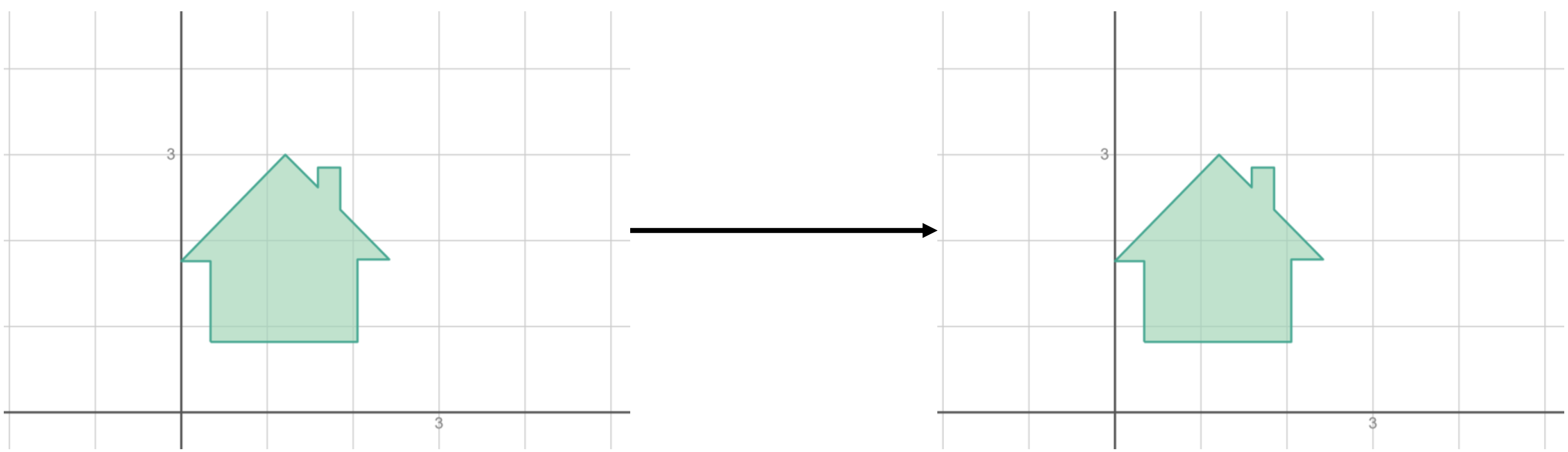
$$\xrightarrow[\begin{bmatrix}1 & 0.5\\0 & 1\end{bmatrix}]{}$$



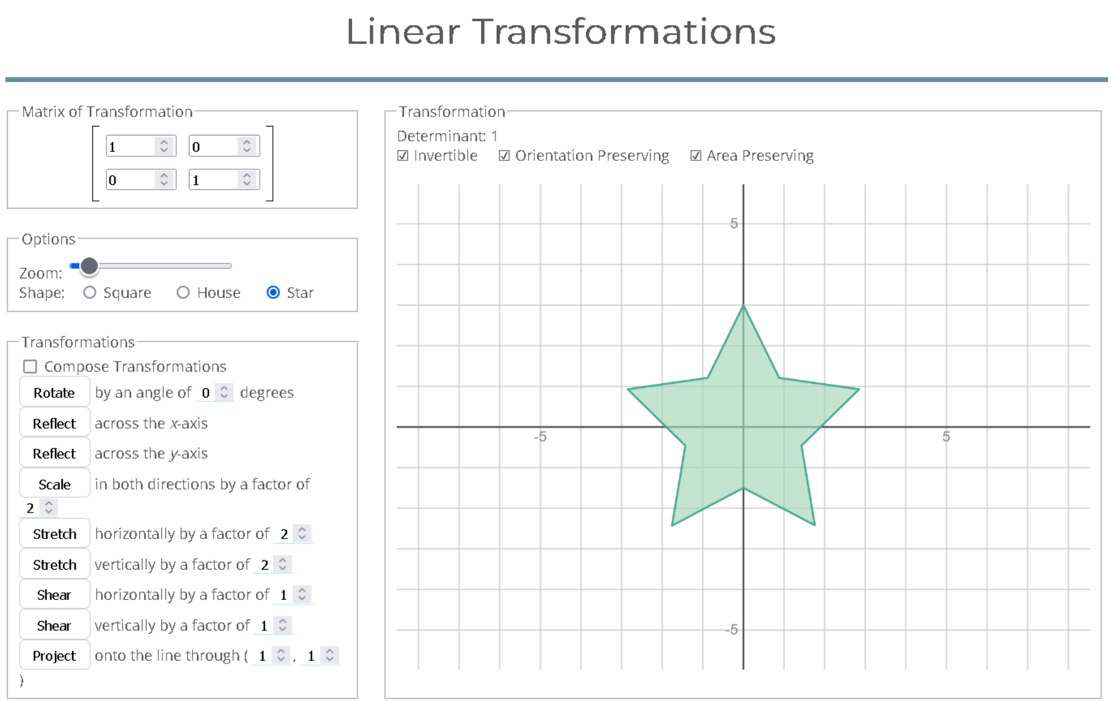
Identity Matrix

- "Doing nothing"

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$$



[Demo] 2D Linear Transformations



https://www.integral-domain.org/lwilliams/Applets/algebra/linearTransformations.php

- Try changing the values of matrix elements.
- Try pressing the transformation buttons.

Quiz 1

- Go to https://www.slido.com/
- Join #**cg-ys**
- Click "Polls"

- Submit your answer in the following format:
  - **Student ID: Your answer**
  - **e.g. 2021123456: 4.0**

- Note that your quiz answer must be submitted **in the above format** to receive a quiz score!

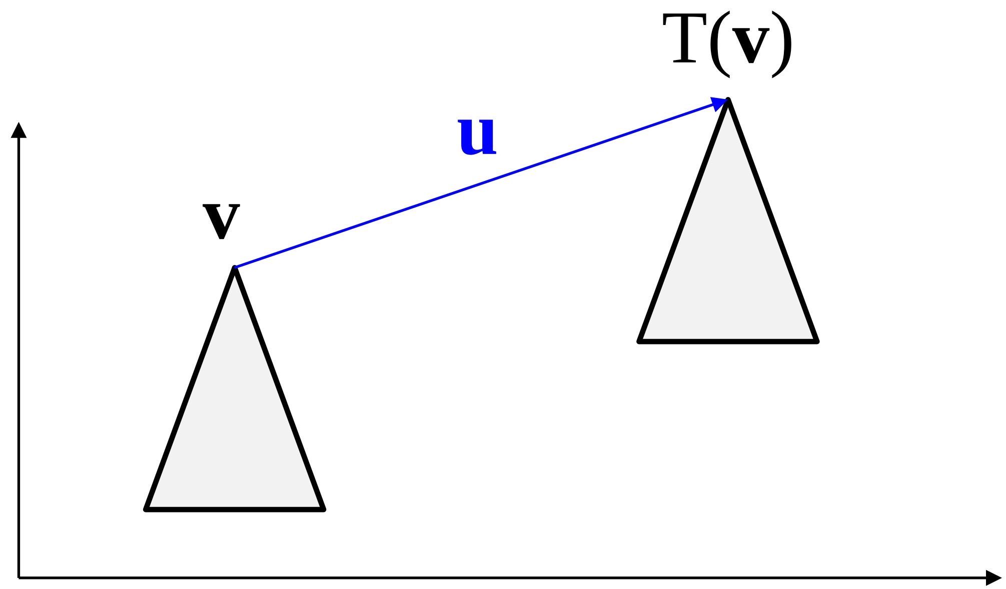
2D Translation

- Translation is the simplest transformation:

$$T(\mathbf{v}) = \mathbf{v} + \mathbf{u}$$

- Inverse:

$$T^{-1}(\mathbf{v}) = \mathbf{v} - \mathbf{u}$$



Is translation linear transformation?

- No, because it cannot be represented using a simple matrix multiplication.
  - Note that a linear transform always maps the zero vector to the zero vector (**0** = **M0** for any **M**).

- We can express translation using vector addition:

$$T(\mathbf{v}) = \mathbf{v} + \mathbf{u}$$

- Combining with linear transformation:

$$T(\mathbf{v}) = M\mathbf{v} + \mathbf{u}$$

- → **Affine transformation**

Let’s check again

- Linear transformation
  - Scaling, rotation, reflection, shearing
  - Represented as matrix multiplications

$$T(\mathbf{v}) = M\mathbf{v}$$

- Translation
  - Not a linear transformation
  - Can be expressed using vector addition

$$T(\mathbf{v}) = \mathbf{v} + \mathbf{u}$$

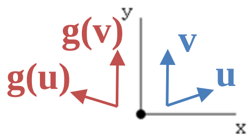
- Affine transformation
  - Combination of linear transformation and translation

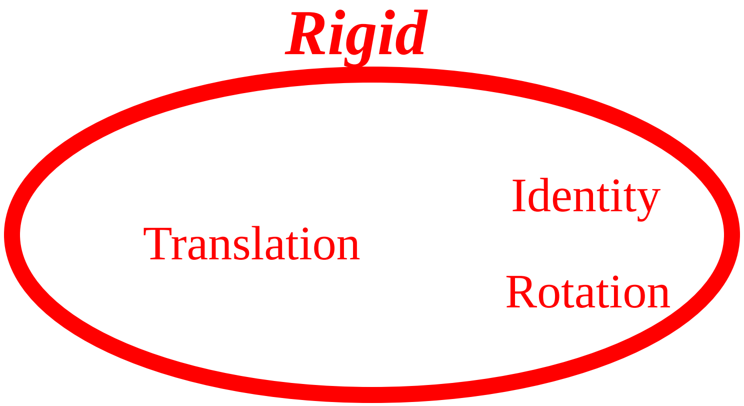
$$T(\mathbf{v}) = M\mathbf{v} + \mathbf{u}$$

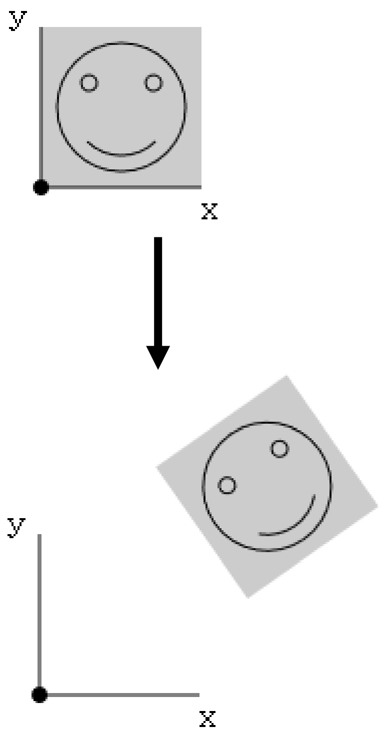
Classes of Transformations

Rigid Transformations

- Preserve distances between all points.
  - $\|g(\mathbf{u}) - g(\mathbf{v})\| = \|\mathbf{u} - \mathbf{v}\|, \ \forall \mathbf{u}, \mathbf{v} \in R^3$ (g: rigid transform map)
- Preserve cross product for all vectors.
  - $g(\mathbf{u}) \times g(\mathbf{v}) = g(\mathbf{u} \times \mathbf{v}), \ \forall \mathbf{u}, \mathbf{v} \in R^3$
  - Reflections do not satisfy this property.



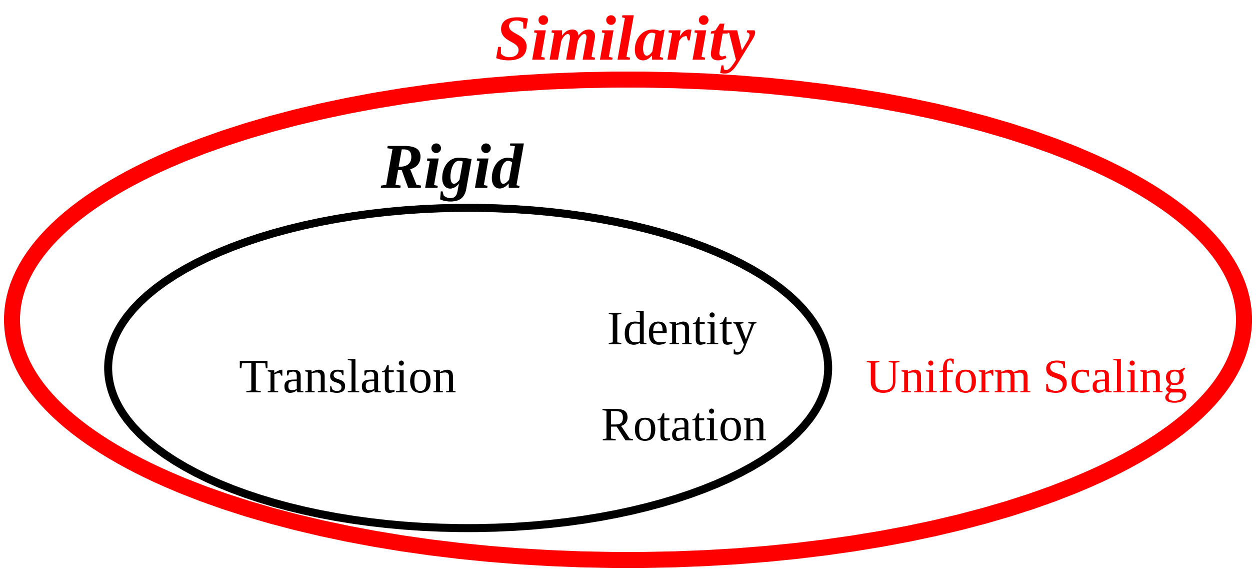


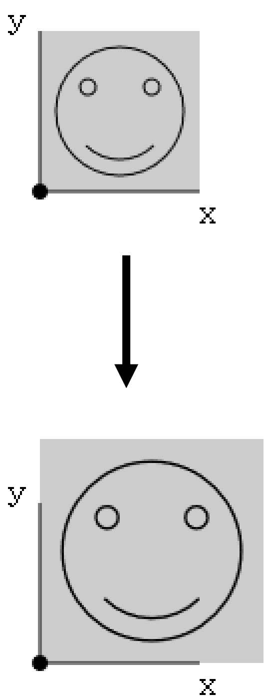


* The diagram is from the slides of Prof. Frédo Durand and Prof. Barbara Cutler (MIT): https://dspace.mit.edu/bitstream/handle/1721.1/86191/6-837-fall-2003/contents/lecture-notes/index.htm

Similarity Transformations

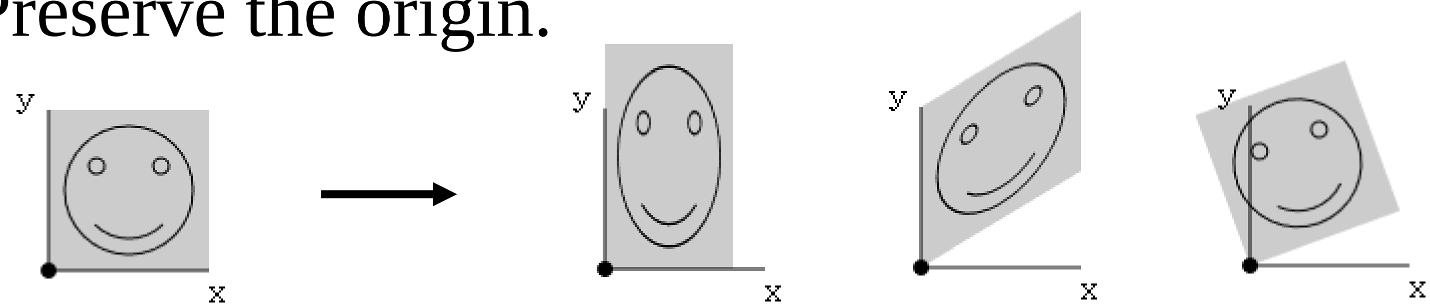
- Preserve angles.
- (This diagram indicates rigid transforms also preserve angles.)

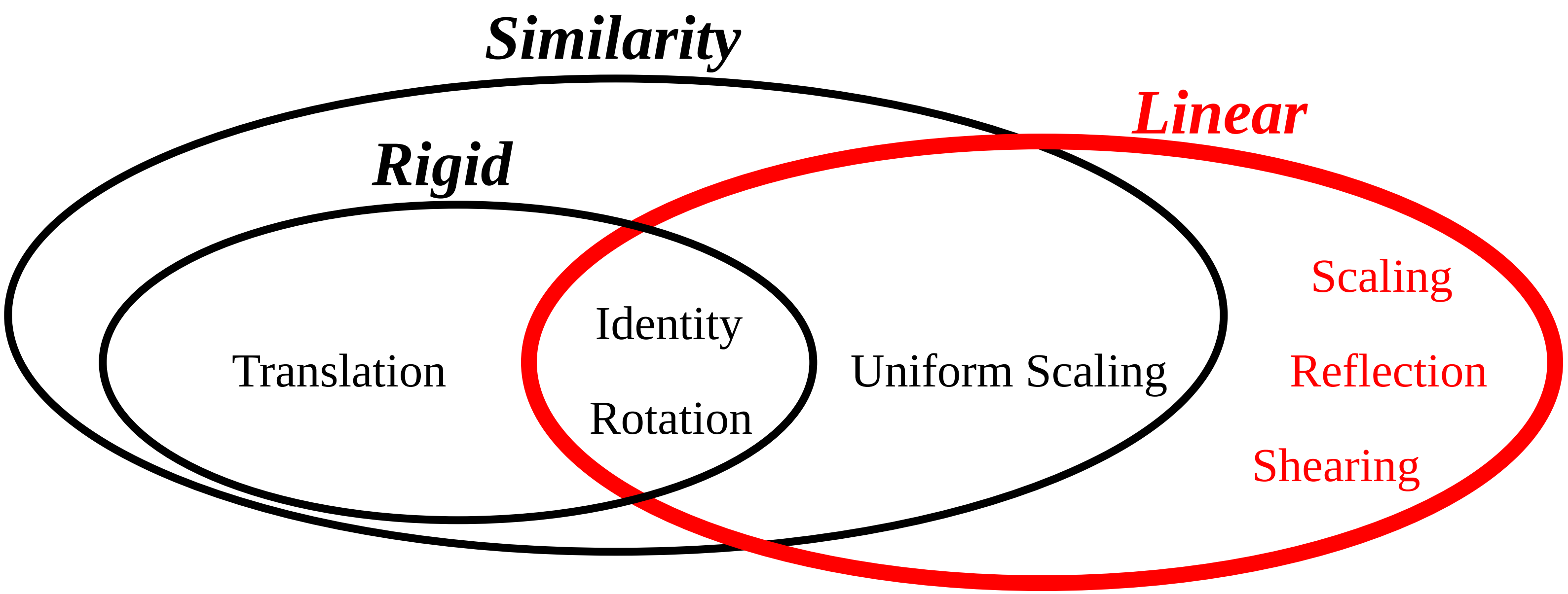




Linear Transformations

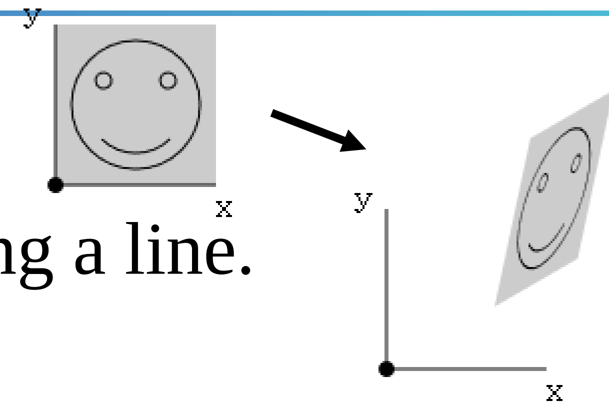
- Preserve the origin.

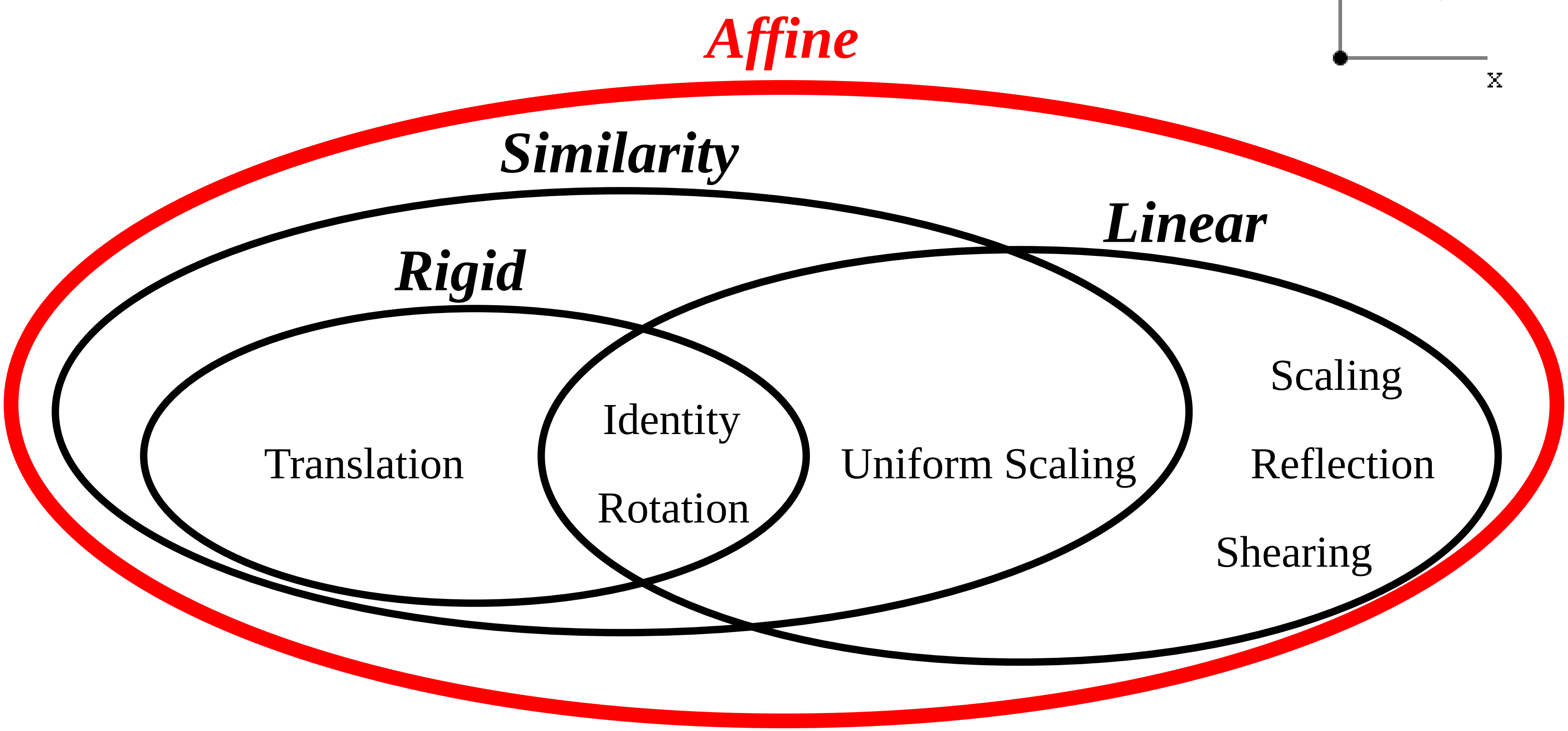




Affine Transformations

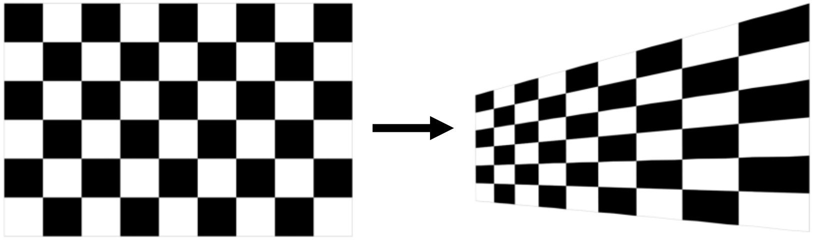
- Preserve parallel lines.
- Preserve ratios of distance along a line.

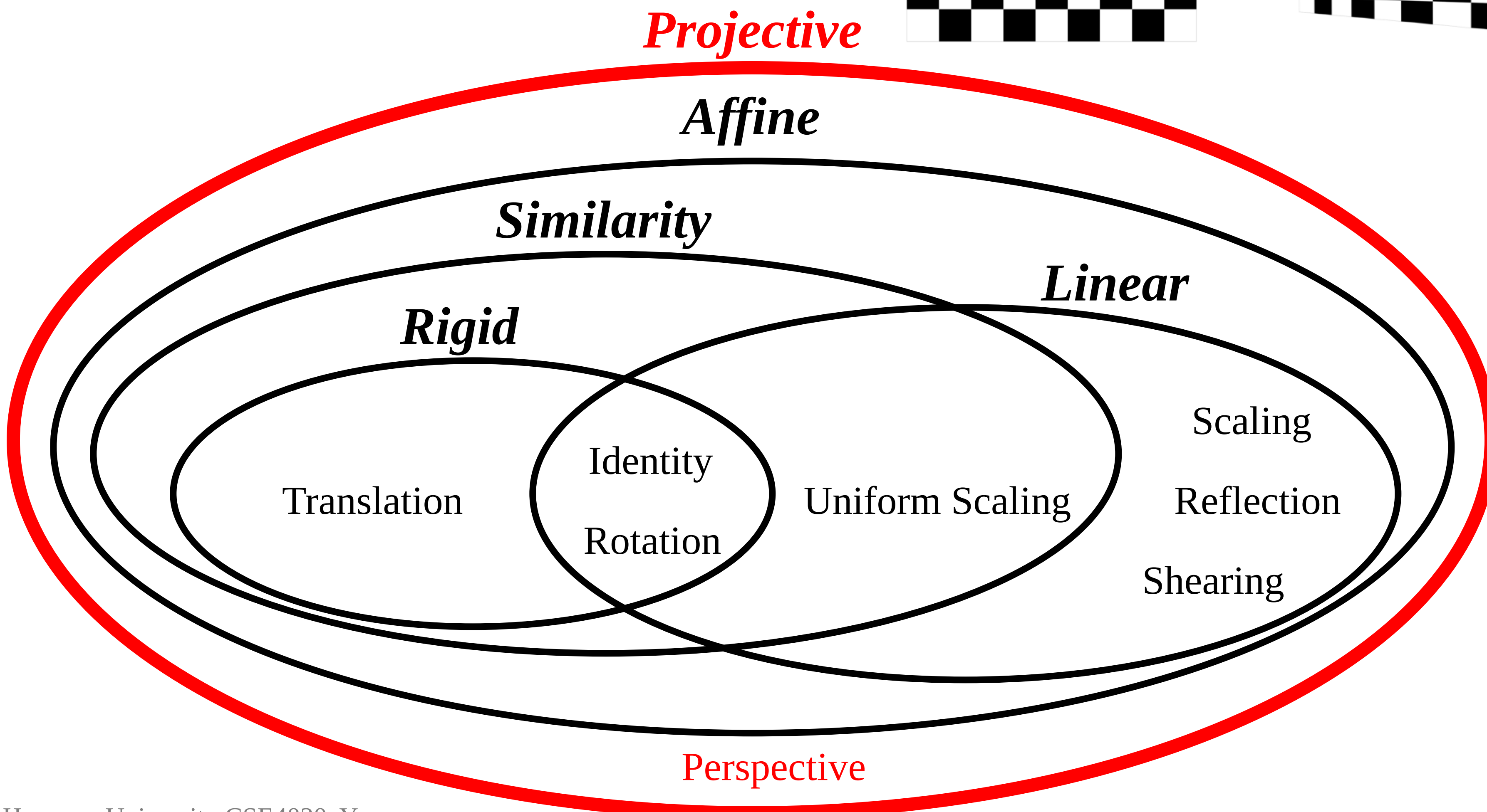




Projective Transformations

- Preserve lines.





Composition of Transformations & Homogeneous Coordinates

Composition of Transformations

- Move an object by T, then move it more by S:

$$\mathbf{p} \to T(\mathbf{p}) \to S(T(\mathbf{p})) = (S \circ T)(\mathbf{p})$$

- **Composing 2D linear transformations** just works by **2x2 matrix multiplication:**

$$T(\mathbf{p}) = M_T\mathbf{p}; S(\mathbf{p}) = M_S\mathbf{p}$$

$$(S \circ T)(\mathbf{p}) = M_S M_T \mathbf{p} = (M_S M_T)\mathbf{p} = M_S(M_T\mathbf{p})$$

* The equation images are from the slides of Prof. Steve Marschner (Cornell University): https://www.cs.cornell.edu/courses/cs4620/2018fa/

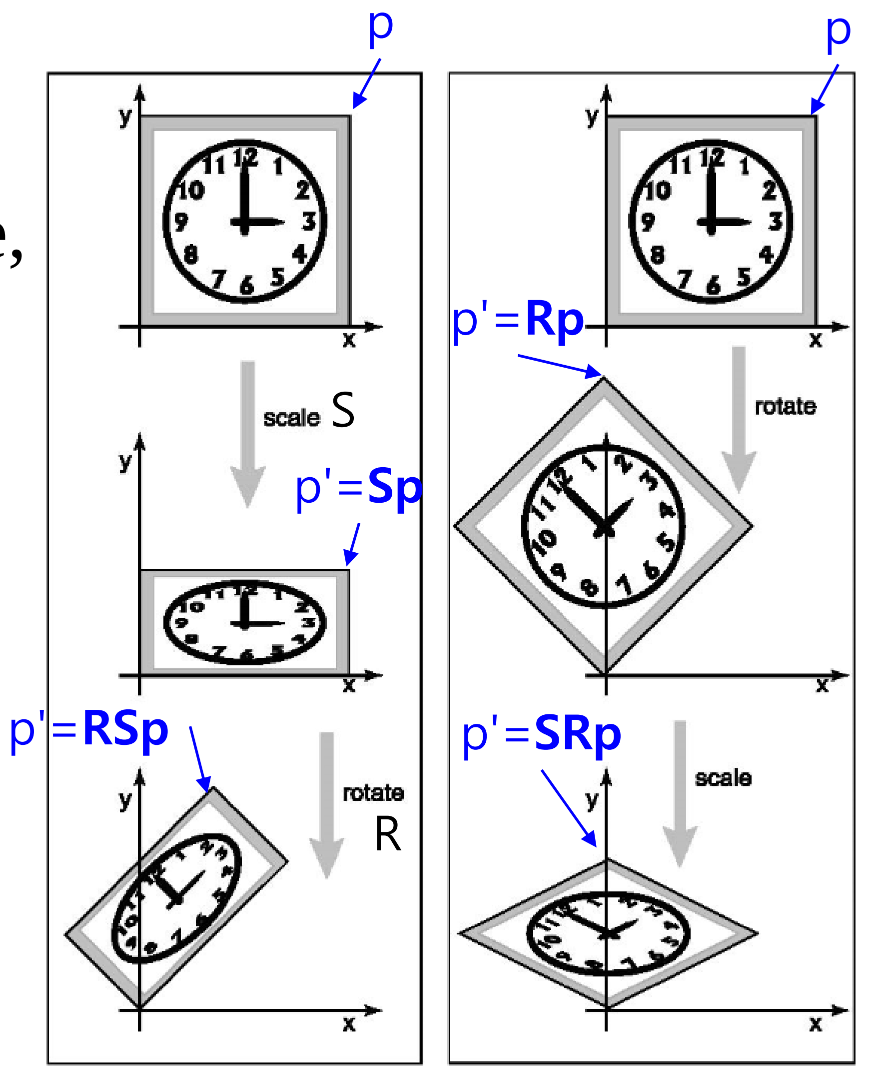
Order Matters!

- Note that matrix multiplication is associative, but **not commutative**.

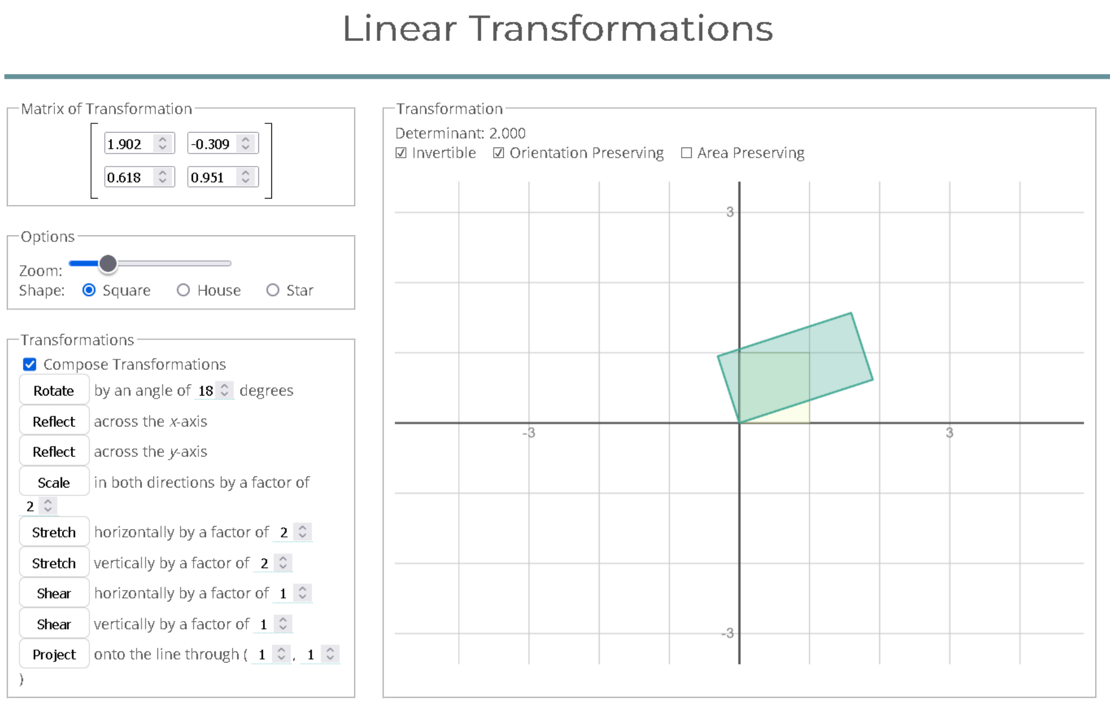
$$(AB)C = A(BC)$$

$$AB \neq BA$$

- As a result, the **order of transforms is very important.**



[Demo] Composition of Linear Transformations



https://www.integral-domain.org/lwilliams/Applets/algebra/linearTransformations.php

- Reset the matrix to the identity matrix (by entering 1 0 0 1).
- Check 'Compose Transformations' button.
- Composites two transforms in different order.

Problems when handling Translation as Vector Addition

- Cannot treat linear transformation (rotation, scale,…) and translation in a consistent manner.

- Composing affine transformations is complicated:

$$T(\mathbf{p}) = M_T\mathbf{p} + \mathbf{u}_T \qquad (S \circ T)(\mathbf{p}) = M_S(M_T\mathbf{p} + \mathbf{u}_T) + \mathbf{u}_S$$

$$S(\mathbf{p}) = M_S\mathbf{p} + \mathbf{u}_S \qquad\qquad = (M_S M_T)\mathbf{p} + (M_S\mathbf{u}_T + \mathbf{u}_S)$$

- We need a cleaner way!
- → **Homogeneous coordinates**

Homogeneous Coordinates

- Key idea: Represent 2D points in 3D coordinate space.

- Extra component *w* for vectors, extra row/column for matrices.
  - For points, always $w = 1$
  - 2D point $[x, y]^T \rightarrow [x, y, 1]^T$.

- 2D linear transformations are represented as:

$$\begin{bmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} ax + by \\ cx + dy \\ 1 \end{bmatrix}$$

Homogeneous Coordinates

- 2D translations are represented as:

$$\begin{bmatrix} 1 & 0 & \boxed{t} \\ 0 & 1 & \boxed{s} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} x \boxed{+ t} \\ y \boxed{+ s} \\ 1 \end{bmatrix}$$

- 2D affine transformations are represented as:

linear part

$$\begin{bmatrix} m_{11} & m_{12} & u_x \\ m_{21} & m_{22} & u_y \\ 0 & 0 & 1 \end{bmatrix}$$

translational part

Homogeneous Coordinates

- **Composing affine transformations** just works by **3x3 matrix multiplication.**

$$T(\mathbf{p}) = M_T\mathbf{p} + \mathbf{u}_T$$
$$S(\mathbf{p}) = M_S\mathbf{p} + \mathbf{u}_S$$

$$T(\mathbf{p}) = \begin{array}{c} \begin{array}{cc} \scriptstyle 2\text{x}2 & \scriptstyle 2\text{x}1 \end{array} \\ \begin{bmatrix} M_T & \mathbf{u}_T \\ 0 & 1 \end{bmatrix} \end{array} \qquad S(\mathbf{p}) = \begin{array}{c} \begin{array}{cc} \scriptstyle 2\text{x}2 & \scriptstyle 2\text{x}1 \end{array} \\ \begin{bmatrix} M_S & \mathbf{u}_S \\ 0 & 1 \end{bmatrix} \end{array}$$

(in block matrix representation)

Homogeneous Coordinates

- **Composing affine transformations** just works by **3x3 matrix multiplication.**

$$(S \circ T)(\mathbf{p}) = \begin{array}{c} \begin{matrix} \text{2x2} & \quad \text{2x1} \end{matrix} \\ \begin{bmatrix} M_S & \mathbf{u}_S \\ 0 & 1 \end{bmatrix} \end{array} \begin{array}{c} \begin{matrix} \text{2x2} & \quad \text{2x1} \end{matrix} \\ \begin{bmatrix} M_T & \mathbf{u}_T \\ 0 & 1 \end{bmatrix} \end{array} \begin{array}{c} \text{2x1} \\ \begin{bmatrix} \mathbf{p} \\ 1 \end{bmatrix} \end{array}$$

$$= \begin{bmatrix} (M_S M_T)\mathbf{p} + (M_S \mathbf{u}_T + \mathbf{u}_S) \\ 1 \end{bmatrix}$$

- The result is the same, but much cleaner.
  - cf. $(S \circ T)(\mathbf{p}) = M_S(M_T\mathbf{p} + \mathbf{u}_T) + \mathbf{u}_S$

$$= (M_S M_T)\mathbf{p} + (M_S \mathbf{u}_T + \mathbf{u}_S)$$

* The equation images are from the slides of Prof. Steve Marschner (Cornell University): https://www.cs.cornell.edu/courses/cs4620/2018fa/

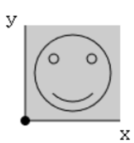
[Demo] Composition of Affine Transformations in Homogeneous Coordinates

Transformation demo

An interactive demo for experimenting with 2D transformation matrix composition.

+ Translate | + Scale | + Rotate | + Shear | Reset

$$T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



https://observablehq.com/@esperanc/transformation-demo

- Add translation and linear transforms in various orders with '+' buttons.
- Drag the slider to see the matrix value change and the shape transform.
- **Note that the last transform added is the first applied transform.**

Summary: Homogeneous Coordinates in 2D

- Use $\mathbf{(x,y,1)^T}$ instead of $(x,y)^T$ for **2D points**
- Use **3x3 matrices** instead of 2x2 matrices **for 2D linear transformations**
- Use **3x3 matrices** instead of vector additions **for 2D translations**

- → We can treat linear transformations and translations **in a consistent manner!**

Quiz 2

- Go to https://www.slido.com/
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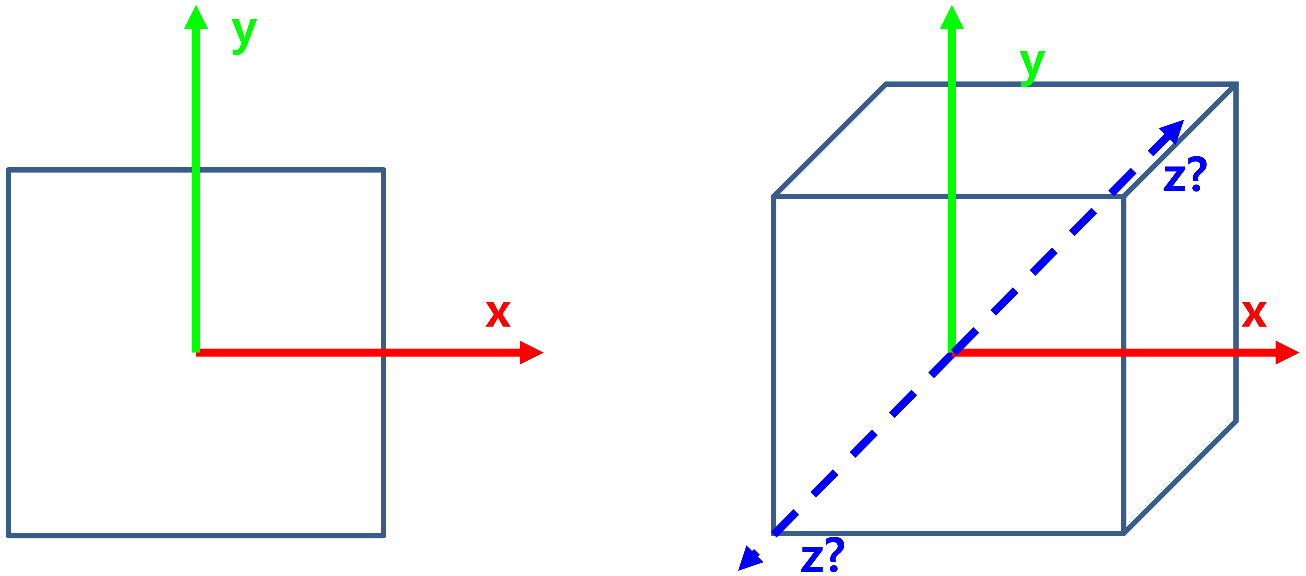
  - **Student ID: Your answer**
  - **e.g. 2021123456: 4.0**

- Note that your quiz answer must be submitted **in the above format** to receive a quiz score!

Two Types of 3D Cartesian Coordinate System

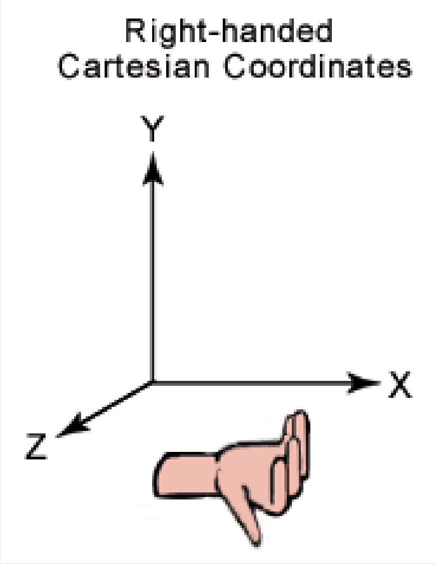
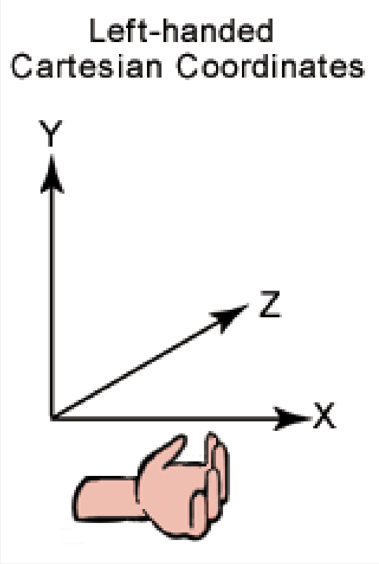
Now, Let's go to the 3D world!



- Coordinate system (좌표계)
  - Cartesian coordinate system (직교좌표계)

Right-Handed and Left-Handed Coordinate Systems

What we're using

		
Positive rotation direction	**counterclockwise** about the axis of rotation 	**clockwise** about the axis of rotation 
Used in…	**OpenGL**, Maya, Houdini, AutoCAD, ... Standard for Physics & Math	DirectX, Unity, Unreal, …

3D Affine Transformations

Point Representations in Cartesian & Homogeneous Coordinate System

	Cartesian coordinate system	Homogeneous coordinate system
A **2D point** is represented as…	$\begin{bmatrix} p_x \\ p_y \end{bmatrix}$	$\begin{bmatrix} p_x \\ p_y \\ \boxed{1} \end{bmatrix}$
A **3D point** is represented as…	$\begin{bmatrix} p_x \\ p_y \\ p_z \end{bmatrix}$	$\begin{bmatrix} p_x \\ p_y \\ p_z \\ \boxed{1} \end{bmatrix}$

Review of Linear Transformations in 2D

- Linear transformations in **2D** can be represented as matrix multiplication of …

2x2 matrix (in Cartesian coordinates)

or

3x3 matrix (in homogeneous coordinates)

$$\begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix} \begin{bmatrix} p_x \\ p_y \end{bmatrix}$$

$$\begin{bmatrix} m_{11} & m_{12} & 0 \\ m_{21} & m_{22} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} p_x \\ p_y \\ 1 \end{bmatrix}$$

Linear Transformations in 3D

- Linear transformations in **3D** can be represented as matrix multiplication of …

3x3 matrix
(in Cartesian coordinates)

or

4x4 matrix
(in homogeneous coordinates)

$$\begin{bmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{bmatrix} \begin{bmatrix} p_x \\ p_y \\ p_z \end{bmatrix} \qquad \begin{bmatrix} m_{11} & m_{12} & m_{13} & 0 \\ m_{21} & m_{22} & m_{23} & 0 \\ m_{31} & m_{32} & m_{33} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} p_x \\ p_y \\ p_z \\ 1 \end{bmatrix}$$

Linear Transformations in 3D

Scale:

3D

$$\mathbf{S_s} = \begin{bmatrix} \mathbf{S}_x & 0 & 0 \\ 0 & \mathbf{S}_y & 0 \\ 0 & 0 & \mathbf{S}_z \end{bmatrix}$$

3D-H

$$\mathbf{S_s} = \begin{bmatrix} \mathbf{S}_x & 0 & 0 & 0 \\ 0 & \mathbf{S}_y & 0 & 0 \\ 0 & 0 & \mathbf{S}_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Shear (in x, based on y,z position):

$$\mathbf{H}_{x,\mathbf{d}} = \begin{bmatrix} 1 & \mathbf{d}_y & \mathbf{d}_z \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

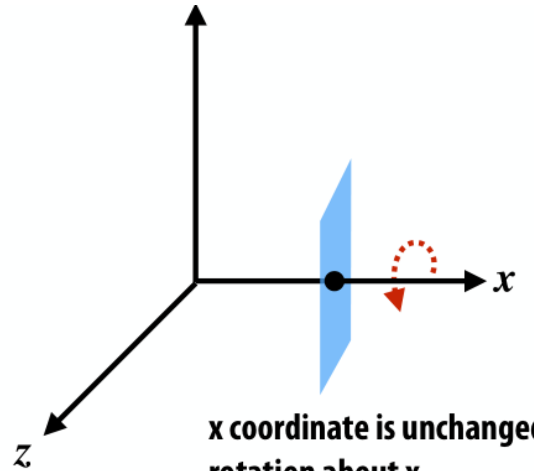
$$\mathbf{H}_{x,\mathbf{d}} = \begin{bmatrix} 1 & \mathbf{d}_y & \mathbf{d}_z & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

* This slide is from the slides of Prof. Kayvon Fatahalian and Prof. Keenan Crane (CMU): http://15462.courses.cs.cmu.edu/fall2015/

Linear Transformations in 3D

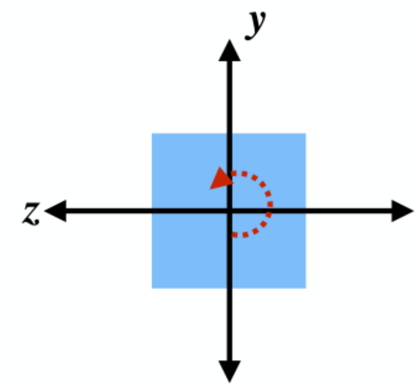
Rotation about x axis:

$$\mathbf{R}_{x,\theta} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$$



x coordinate is unchanged by rotation about x

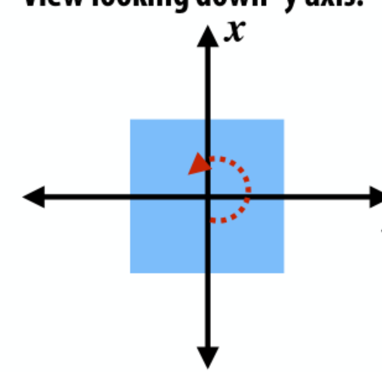
View looking down -x axis:



Rotation about y axis:

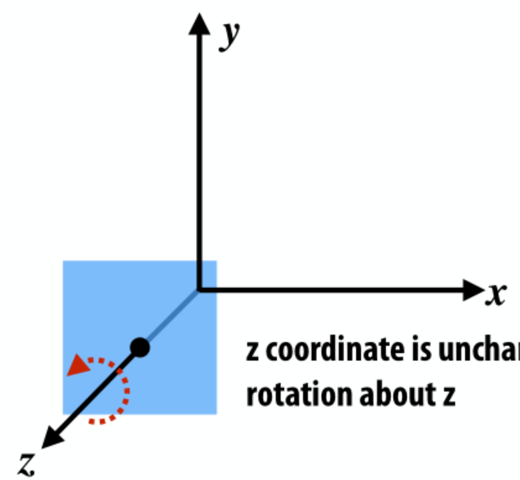
$$\mathbf{R}_{y,\theta} = \begin{bmatrix} \cos\theta & 0 & \boxed{}\sin\theta \\ 0 & 1 & 0 \\ \boxed{-}\sin\theta & 0 & \cos\theta \end{bmatrix}$$

View looking down -y axis:



Rotation about z axis:

$$\mathbf{R}_{z,\theta} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



z coordinate is unchanged by rotation about z

* This slide is from the slides of Prof. Kayvon Fatahalian and Prof. Keenan Crane (CMU): http://15462.courses.cs.cmu.edu/fall2015/

Review of Translations in 2D

- Translations in **2D** can be represented as …

Vector addition
(in Cartesian coordinates)

$$\begin{bmatrix} p_x \\ p_y \end{bmatrix} + \begin{bmatrix} u_x \\ u_y \end{bmatrix}$$

Matrix multiplication of
3x3 matrix
(in homogeneous coordinates)

$$\begin{bmatrix} 1 & 0 & u_x \\ 0 & 1 & u_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} p_x \\ p_y \\ 1 \end{bmatrix}$$

Translations in 3D

- Translations in **3D** can be represented as …

Vector addition
(in Cartesian coordinates)

$$\begin{bmatrix} p_x \\ p_y \\ p_z \end{bmatrix} + \begin{bmatrix} u_x \\ u_y \\ u_z \end{bmatrix}$$

Matrix multiplication of
4x4 matrix
(in homogeneous coordinates)

$$\begin{bmatrix} 1 & 0 & 0 & u_x \\ 0 & 1 & 0 & u_y \\ 0 & 0 & 1 & u_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} p_x \\ p_y \\ p_z \\ 1 \end{bmatrix}$$

Review of Affine Transformations in 2D

- In homogeneous coordinates, **2D** affine transformations can be represented as multiplication of **3x3 matrix**:

linear part

translational part

$$\begin{bmatrix} m_{11} & m_{12} & u_x \\ m_{21} & m_{22} & u_y \\ 0 & 0 & 1 \end{bmatrix}$$

Affine Transformations in 3D

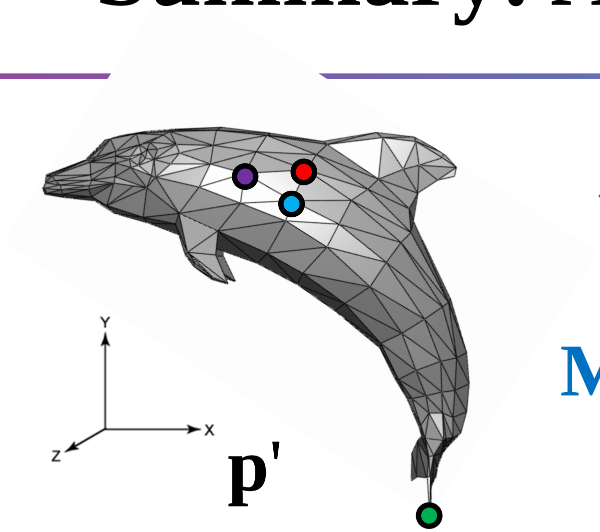
- In homogeneous coordinates, **3D** affine transformations can be represented as multiplication of **4x4 matrix**:

linear part

translational part

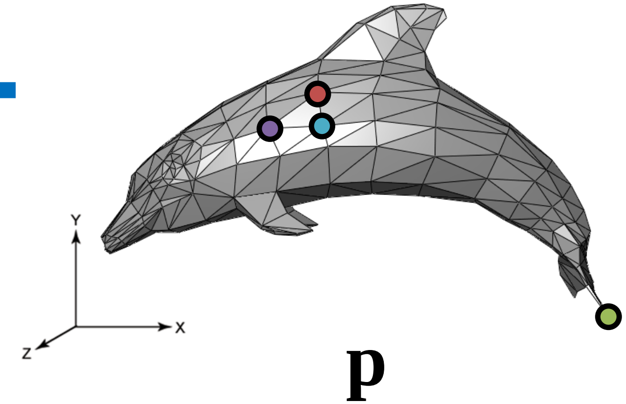
$$\begin{bmatrix} m_{11} & m_{12} & m_{13} & u_x \\ m_{21} & m_{22} & m_{23} & u_y \\ m_{31} & m_{32} & m_{33} & u_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Summary: Affine Transformation



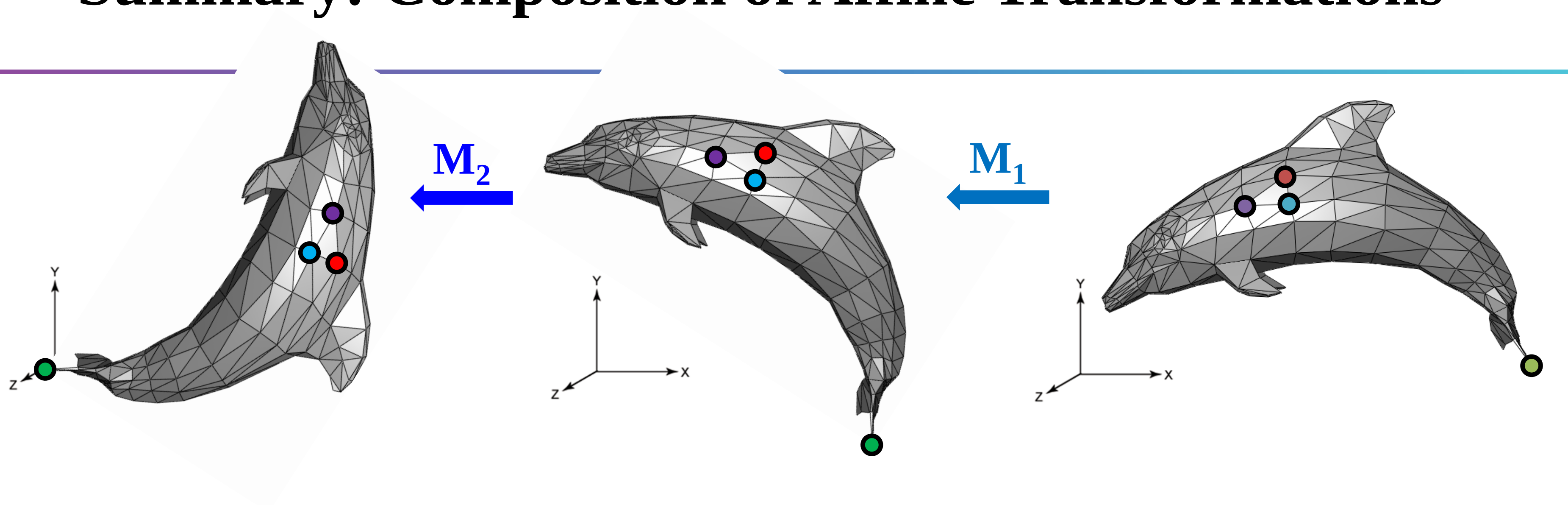
Affine transformation ←

$$\mathbf{M} = \begin{bmatrix} m_{11} & m_{12} & m_{13} & u_1 \\ m_{21} & m_{22} & m_{23} & u_2 \\ m_{31} & m_{32} & m_{33} & u_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



$$\mathbf{p}_1' \leftarrow \mathbf{M}\,\mathbf{p}_1$$
$$\mathbf{p}_2' \leftarrow \mathbf{M}\,\mathbf{p}_2$$
$$\mathbf{p}_3' \leftarrow \mathbf{M}\,\mathbf{p}_3$$
$$\vdots \qquad \vdots \quad \vdots$$
$$\mathbf{p}_N' \leftarrow \mathbf{M}\,\mathbf{p}_N$$

Summary: Composition of Affine Transformations



$$
\begin{aligned}
p_1' &\leftarrow M_2 \, M_1 \, p_1 \\
p_2' &\leftarrow M_2 \, M_1 \, p_2 \\
p_3' &\leftarrow M_2 \, M_1 \, p_3 \\
\vdots \quad & \quad\;\; \vdots \quad\; \vdots \quad\; \vdots \\
p_N' &\leftarrow M_2 \, M_1 \, p_N
\end{aligned}
$$

Lab Session

- Now, let's start the lab today.